Gabor Frames, the Zak Transform and Balian-Low

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Abstract

The Zak Transform, introduced by Gelfand and rediscovered by Zak, has proven to be a transformational tool in studying the Gabor frames ubiquitous in time-frequency analysis. The utility of the Zak transform is shown to arise from the conjugacy equivalence of the Gabor frame operator to multiplication by the square magnitude of the Zak transform. The Zak transform can also be elucidated as a flexible variant of the Poisson summation formula, hence providing a link between the theory of Gabor frames and more classical harmonic analysis. This theory is employed in the study of Balian-Low type uncertainty principles and in the development of the alternate uncertainty principle on Wiener Amalgam Space.

I. Gabor Systems

Gabor systems, alternately Weyl-Heisenberg systems, were studied in signal processing as a type of localized discrete Fourier transform and arose naturally in work on operator algebras by Von Neumann. They are deeply connected to both quantum mechanics and signal processing and provide a myriad of fascinating and difficult questions in pure mathematics. We focus our attention on so called regular Gabor frames, those that tile the time-frequency "phase plane" into a repeating lattice.

Definition 1. *(Gabor systems). Given a generating (window) function $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}_{>0}$, the corresponding (regular) Gabor system is given by:*

$$\mathcal{G}_{a,b}(g) = \{M_{mb}T_{na}g | m, n \in \mathbb{Z}\} \tag{1}$$

where $T_\tau(g)(t) = g(t-\tau)$ is the translation operator on $L^2(\mathbb{R})$ and $M_\omega(g)(t) = e^{2\pi i\omega t}g(t)$ is the modulation operator.

We shall see that the following transform operator plays a similar role to that of the Fourier transform in understanding complex exponential bases for $L^2(\mathbb{R})$. Moreover, it plays a fundamental role in developing the corresponding uncertainty results for Gabor systems.

Definition 2. *(The Zak transform). For a fixed dilation parameter $\lambda > 0$ and for $(x, \xi) \in \mathbb{R} \times \mathbb{T}$ the Zak transform of a function $f \in C_c(\mathbb{R})$ is defined*

$$Z_\lambda : C_c(\mathbb{R}) \to C(\mathbb{R} \times \mathbb{T})$$
$$Z_\lambda(f)(x, \xi) = \lambda^{-1/2} \sum_{n\in\mathbb{Z}} f(\lambda(x-n))e^{2\pi i n\xi} \tag{2}$$

The continuity of the Zak transform for $f \in C_c(\mathbb{R})$ follows immediately from the fact that all but finitely many terms in the sum are zero. We may extent this definition to $f \in L^2(\mathbb{R})$ (and indeed beyond to distributions) by noting that

$$Z_\lambda(f)(x+m, \xi) = e^{-2\pi i m\xi} Z_\lambda(f)(x, \xi) \tag{3}$$

This motivates us to define the fundamental domain of the image of the Zak transform as $Q = [0, 1) \times \mathbb{T}$, and indeed we shall prove

Lemma 1. *Given $\lambda > 0$, the Zak transform Z_λ is a unitary map of $L^2(\mathbb{R}) \to L^2(Q)$.*

Proof. Consider $\lambda = 1$. Given $f \in L^2(\mathbb{R})$, consider $\{F_k\}_{k\in\mathbb{Z}}$

$$F_k(x, \xi) = f(x-k)e^{2\pi i k\xi} \tag{4}$$

Clearly each F_k belongs to $L^2(Q)$ since $\overline{Q}$ is compact. Then note that

$$\sum_{k\in\mathbb{Z}} ||F_k||^2_{L^2(Q)} = \sum_{k\in\mathbb{Z}} \int_0^1 \int_0^1 |F_k(x,\xi)|^2 dx d\xi = \sum_{k\in\mathbb{Z}} \int_0^1 |f(x-k)|^2 dx = ||f||^2_{L^2(\mathbb{R})} < \infty \tag{5}$$

Which is more, note that for $k \neq j$ we have

$$\langle F_k, F_j \rangle_{L^2(\mathbb{R})} = \int_0^1 f(x-k)\overline{f(x-j)} \left(\int_0^1 e^{2\pi i(k-j)\xi} d\xi \right) dx = 0 \tag{6}$$

Hence

$$||Z_1(f)(x,\xi)||^2_{L^2(Q)} = ||\sum_{k\in\mathbb{Z}} F_k||^2_{L^2(Q)} = \sum k \in \mathbb{Z} ||F_k||_{L^2(Q)} = ||f||^2 \tag{7}$$

This concludes the proof for the case $\lambda = 1$. For the general case, note that $Z_\lambda(f) = Z_1(D_{\lambda^{-1}} f)$ is a composition of unitary operators. □

Remark The Zak transform is precisely the Fourier series on $\mathbb{T}$ corresponding to the the coefficients $\{T_{na}g\}_{n\in\mathbb{Z}}$, hence $g(x-na) = (\mathcal{F}_\xi^{-1} Z_a(g))(x,n) = (\mathcal{F}_\xi^{-1} Z_a(g))(x-na,0)$. The latter equality is immediate from the fact that x and n appear only as $x - na$.

We will also need the following

Lemma 2. *For $\lambda > 0$ and $f \in L^2(\mathbb{R})$ the following hold*

(i) If f is continuous and for some $C > 0$ f satisfies

$$|f(x)| \leq \frac{C}{1+|x|^2} \qquad \forall x \in \mathbb{R} \tag{8}$$

, then $Z_\lambda f$ is continuous on $\mathbb{R}^2$.

(ii) If $Z_\lambda f$ is continuous on $\mathbb{R}^2$ then there exists $(x,\xi) \in \mathbb{R}^2$ such that $Z\lambda(f)(x,\xi) = 0$.

The proof of of (ii) would take us too far afield but can be found in [9]. For (i) we observe that

$$\begin{aligned}
&|Z_\lambda(f)(x,\xi) - Z_\lambda(f)(y,\omega)| = \lambda^{-1/2} |\sum_{n\in\mathbb{Z}} f(\lambda(x-na))e^{2\pi i\xi} - f(\lambda(y-na))e^{2\pi i\omega}| \\
&\leq \lambda^{-1/2} \sum_{n\in\mathbb{Z}} |f(\lambda(x-na))e^{2\pi i(\xi-\omega)} - f(\lambda(y-na))| \\
&= \lambda^{-1/2} \sum_{n\in\mathbb{Z}} |f(\lambda(x-na))e^{2\pi i(\xi-\omega)} - f(\lambda(y-na))e^{2\pi i(\xi-\omega)} + f(\lambda(y-na))e^{2\pi i(\xi-\omega)} - f(\lambda(y-na))| \\
&\leq \lambda^{-1/2} \sum_{n\in\mathbb{Z}} |f(\lambda(x-na)) - f(\lambda(y-na))| + \lambda^{-1/2} \sum_{n\in\mathbb{Z}} \frac{C}{1+\lambda^2(y-na)^2} |e^{2\pi in(\xi-\omega)} - 1|
\end{aligned} \tag{9}$$

Given $\epsilon > 0$ the fact that$f(x) \leq C/(1+|x|^2)$ implies that N can be chosen large enough so that the tail of the first sum is less than $\epsilon/4$. The first N terms in the first sum can then be made less than $\epsilon/4$ using the continuity of f. Finally, the second sum can be made to be less than $\epsilon/2$ by employing the continuity of the complex exponential. □

The deep connection between Gabor frames and the Zak transform is made clear by the following theorem, which can be extended to the case $ab \in \mathbb{Q}$ (for ab irrational other tools than the Zak transform are needed).

Theorem 1. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ with $ab = 1$. Then*

(i) $\mathcal{G}_{a,b}(g)$ is complete in $L^2(\mathbb{R})$ if and only if $Z_a(g) \neq 0$ almost everywhere.

(ii) $\mathcal{G}_{a,b}(g)$ is a Bessel sequence with bound B if and only if $|Z_a(g)|^2 \leq B$ almost everywhere.

(iii) $\mathcal{G}_{a,b}(g)$ is a Riez basis for $L^2(\mathbb{R})$ with bounds A and B if and only if $A \leq |Z_a(g)|^2 \leq B$ almost everywhere.

(iv) $\mathcal{G}_{a,b}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|Z_a(g)| = 1$ almost everywhere.

We note the following corollary:

Corollary 1. *Let g be a continuous function with compact support, then*

1) $\mathcal{G}_{a,b}(g)$ cannot be an orthonormal basis for $L^2(\mathbb{R})$

2) $\mathcal{G}_{a,b}(g)$ cannot be a Riesz basis for $L^2(\mathbb{R})$

Proof. We follow [5]. Let $V \subset L^2(\mathbb{R})$ be given as

$$V = \{f \in L^2(\mathbb{R}) | Z_a(f) \text{ is bounded}\} \tag{10}$$

The bounded functions are dense in $L^2(Q)$ and Lemma 1 tells us that $L^2(Q)$ is the image of $L^2(\mathbb{R})$ under the unitary transformation Z_a, hence we obtain that V is dense in $L^2(\mathbb{R})$. Now suppose that $f \in V$, then first note that since $ab = 1$ we have (treating the Zak transform as an operator)

$$\begin{aligned}
Z_a M_{mb} T_{na} g &= Z_a e^{2\pi imbx} g(x-na) \\
&= a^{-1/2} \sum_{k\in\mathbb{Z}} e^{2\pi imba(x-k)} g(a(x-k)-na) e^{2\pi ik\xi} \\
&= a^{-1/2} \sum_{k\in\mathbb{Z}} e^{2\pi imx} e^{2\pi ik\xi} g(a(x-k-n)) \\
&= e^{2\pi imx} e^{2\pi in\xi} a^{-1/2} \sum_{l\in\mathbb{Z}} e^{2\pi il\xi} g(a(x-l)) = M_{m,n} Z_a g
\end{aligned} \tag{11}$$

Where $M_{m,n} = e^{2\pi imx}e^{2\pi in\xi}$ provides a Fourier basis for Q. Using this and the fact that Z_a is unitary we obtain

$$\begin{aligned}\langle f, M_{mb}T_{na}g\rangle_{L^2(\mathbb{R})} &= \langle Z_a f, Z_a M_{mb}T_{na}g\rangle_{L^2(Q)} = \langle M_{m,n}Z_a g\rangle_{L^2(Q)} \\ &= \langle Z_a(f)\overline{Z_a(g)}, M_{m,n}\rangle_{L^2(Q)}\end{aligned} \tag{12}$$

Assume $Z_a(g) \neq 0$ almost everywhere, then if $f \neq 0$ we have that $Z_a(f)\overline{Z_a(g)}$ is not the zero function, hence since $M_{m,n}$ is an orthonormal basis for Q there exists p and q such that $\langle Z_a(f)\overline{Z_a(g)}, M_{p,q}\rangle_{L^2(Q)} \neq 0$. From this and (12) we conclude that $\mathcal{G}_{a,b}(g)$ is complete in $L^2(\mathbb{R})$. On the other hand assume that $Z_a(g)$ is zero on a set of positive measure B, then choose f such that $Z_a f = \mathbb{1}_{Q\setminus B}$. In this case $\langle f, M_{mb}T_{na}g\rangle_{L^2(\mathbb{R})} = 0$ for all $m, n \in \mathbb{Z}$, hence $\mathcal{G}_{a,b}(g)$ cannot be complete in $L^2(\mathbb{R})$. In order to prove *(ii)-(iv)* we note that if $F \in L^2(\mathbb{R})$ then since $M_{m,n}$ is an orthonormal basis for $L^2(Q)$

$$\int_Q |F\overline{Z_a g}|^2 = \sum_{m,n\in\mathbb{Z}} |\langle F\overline{Z_a g}, M_{m,n}\rangle_{L^2(Q)}|^2 = \sum_{m,n\in\mathbb{Z}} |\langle F, M_{m,n}Z_a g\rangle_{L^2(Q)}|^2 \tag{13}$$

By the assumption of *(ii)* the leftmost integral satisfies

$$\int_Q |F\overline{Z_a g}|^2 \leq B||F||_{L^2(Q)} \tag{14}$$

Moreover this inequality holds for all F in $L^2(Q)$ only when the assumption of *(ii)* holds also. Thus we obtain, again using (12), that

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle F, Z_a M_{mb}T_{na}g\rangle_{L^2(Q)}|^2 \leq B||F||_{L^2(Q)} \forall F \in L^2(Q) &\iff |Z_a g|^2 \leq B \\ \sum_{m,n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle_{L^2(\mathbb{R})}|^2 \leq B||f||_{L^2(\mathbb{R})} \forall f \in L^2(\mathbb{R}) &\iff |Z_a g|^2 \leq B\end{aligned} \tag{15}$$

Where in the second line we set $f = Z_a^{-1}F$ and used unitarity several times. This is of course the definition of a Bessel sequence, so *(ii)* is proved. The proof of *(iii)* is identical only with lower bound included, and *(iv)* follows immediately from *(iii)* and the well known fact that a Gabor system with $ab = 1$ is a frame if and only if it is a Riesz basis.□

II. Spectral Theory

Definition 3. *Let $F = \{f_\alpha\}_{\alpha\in I}$ be a frame for $L^2(\mathbb{R})$. Then the corresponding frame operator S_F is*

$$\begin{aligned} S_F : L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ S_F(h) = \sum_{\alpha\in I} \langle f\alpha, h\rangle_{L^2(\mathbb{R})} f_\alpha &\qquad \textit{for } h \in L^2(\mathbb{R})\end{aligned} \tag{16}$$

With this definition Theorem 1 can be reformulated in the language of spectral theory by observing that the frame operator for regular Gabor frames $\mathcal{G}_{a,b}(g)$ with $ab = 1$ S_g is precisely conjugate to the multiplication operator $T : L^2(Q) \to L^2(Q)$ with $Tf = |Z_a g|^2 \cdot f$. The results in Theorem 1 follow, but in fact this result (following [7]) allows a full characterization of the spectral calculus of frame operators for regular Gabor systems.

Theorem 2. *Let $g \in L^2(\mathbb{R})$ and $ab = 1$. Then*

$$Z_a S_g f = |Z_a g|^2 Z_a f \tag{17}$$

Where S_g is the frame operator for $\mathcal{G}_{a,b}(g)$.

$$S_g(h) = \sum_{m,n\in\mathbb{Z}} \langle h, M_{mb}T_{na}g\rangle_{L^2(\mathbb{R})} M_{mb}T_{na}g \tag{18}$$

Proof. We again employ the crucial operator identity $Z_a M_{mb}T_{na} = M_{m,n}Z_a$ for $ab = 1$, as well as the unitarity of Z_a. Namely,

$$\begin{aligned} Z_a S_g f &= Z_a \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle_{L^2(\mathbb{R})} M_{mb}T_{na}g \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle_{L^2(\mathbb{R})} Z_a M_{mb}T_{na}g \\ &= \sum_{m,n\in\mathbb{Z}} \langle Z_a f, M_{m,n}Z_a g\rangle_{L^2(Q)} M_{m,n}Z_a g \\ &= |Z_a g|^2 \sum_{m,n\in\mathbb{Z}} \langle Z_a, M_{m,n}\rangle_{L^2(Q)} M_{m,n} \\ &= |Z_a g|^2 Z_a f \end{aligned} \tag{19}$$

Where in the last step we use the fact that $M_{m,n}$ is an orthonormal basis for $L^2(Q)$. $\square$
We therefore conclude that $S_g = Z_a^{-1} X_{|Z_a|^2 g} Z_a$ where X_α is the multiplication operator by α on $L^2(Q)$. This is a beautiful result in its own right, but it also provides an immediate proof of Theorem 1. Moreover, it gives the optimal frame bounds for $\mathcal{G}_{a,b}(g)$ as

$$\begin{aligned} A_{opt} &= \text{essinf}_{(x,\xi)\in Q} |Z_a g(x,\xi)|^2 \\ B_{opt} &= \text{esssup}_{(x,\xi)\in Q} |Z_a g(x,\xi)|^2 \end{aligned} \tag{20}$$

Theorem 2 also provides convenient accesss to the holomorphic functional calculus for the frame operator S_g, easily providing both the inverse and unique positive definite square root of the frame operator as

$$\begin{aligned} Z_a S_g^{-1} &= |Z_a g|^{-2} Z_a f \\ Z_a S_g^{\pm\frac{1}{2}} &= |Z_a g|^{\pm 1} Z_a f \end{aligned} \tag{21}$$

This result can be applied for example to compute the canonical dual window $\gamma^0 = S_g^{-1} g$ as

$$Z_a \gamma_0 = |Z_a g|^{-2} Z_a g = \frac{1}{\overline{Z_a g}} \tag{22}$$

Moreover the frame reconstruction formula for $\mathcal{G}_{a,b}(g)$ yields

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, S_g^{-1/2} g_{m,n} \rangle_{L^2(\mathbb{R})} S_g^{-1/2} g_{m,n} = \sum_{m,n\in\mathbb{Z}} \langle Z_a f, |Z_a g|^{-1} Z_a g_{m,n} \rangle_{L^2(Q)} Z_a^{-1} |Z_a g|^{-1} Z_a g_{m,n} \tag{23}$$

For each $f \in L^2(\mathbb{R})$.

III. Balian-Low

The Balian Low theorem is an uncertainty principle concerning Gabor systems with $ab = 1$. The theorem was stated independently by Balian [1] and Low [10]. Both proofs contained a gap that was resolved by Daubechies, Coifman, and Semmes in [6]. The original proofs employed the Zak transform and *(ii)* from Lemma 2, but an alternate proof was given by Battle in [2] employing operator theory and relating the result back to the classical uncertainty principle. The statement of the theorem is

Theorem 3. *Let g be such $\mathcal{G}_{a,b}(g)$ be a Gabor system with $ab = 1$ that forms an orthonormal basis for $L^2(\mathbb{R})$. Then*

$$\left(\int_{-\infty}^{\infty} |xg(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} |\xi\hat{g}(\xi)|^2 d\xi\right) = +\infty \tag{24}$$

A natural question, investigated in [4], is to determine whether the theorem is sharp in the sense that the weights $|x|^2$ and $|\xi|^2$ cannot be replaced by something smaller. The answer is almost certainly no – indeed in [4] the authors construct a generating function $g \in L^2(\mathbb{R})$ satisfying the following

Theorem 4. *Let $\epsilon > 0$. There is a constructible $g \in L^2(\mathbb{R})$ with the property that $\mathcal{G}_{a,b}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$ and such that*

$$\int_{-\infty}^{\infty} |g(x)|^2 \frac{1+|x|^2}{\log^{1+\epsilon}(2+|x|)} dx < \infty \tag{25}$$

and

$$\int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 \frac{1+|\xi|^2}{\log^{1+\epsilon}(2+|\xi|)} d\xi < \infty \tag{26}$$

Proof. Following [4] we consider the case $a = b = 1$ and denote Z_1 as Z. Owing to Theorem 1 it is necessary to consider functions of the form

$$Z(g)(x,\xi) = h(x,\xi) = e^{2\pi i H(x,\xi)} \tag{27}$$

In order to insure that $Z(g)$ is well defined on Q we will require that $H(1,\xi) = H(0,\xi) + k_\xi$ where $k_\xi \in \mathbb{Z}$. Note however that Lemma 2 implies that there do not exist any continuous functions satisfying this constraint, since if the Zak transform of a function $g \in L^2(\mathbb{R})$ is continuous it must have a zero. Thus h must have at least one point of singularity. The authors of [4] construct such a function with exactly one point of singularity and explain that they hope this will give rise to good decay properties for g and $\hat{g}$. They introduce the following

Definition 4. *(1) Let $\phi \in C^\infty(\mathbb{R})$ be such that*

$$\begin{aligned} \phi(x) &= -1, && \text{for } x \in (-\infty, 0], \\ \phi(x) &= 0, && \text{for } x \in [1, \infty), \\ \phi(x) &\in [-1, 0], && \text{for all } x \in \mathbb{R} \end{aligned} \tag{28}$$

(2) Let $\psi(x) = \mathbb{1}_{[0,\infty)} x^a$ where $a > 0$ is fixed
(3) Given $0 < \epsilon < \frac{1}{8}$ let $\gamma \in C^\infty(\mathbb{R})$ be a function satisfying

$$supp(\gamma) \subseteq [-2\epsilon, 2\epsilon], \qquad \gamma(x) = 1 \text{ for } x \in [-\epsilon, \epsilon] \tag{29}$$

and

$$\gamma(x) \in [0, 1] \text{ for all } x \in \mathbb{R} \tag{30}$$

Given the above definition, the authors of [4] employ the following construction

Lemma 3. *There exists a function $H : [-\frac{1}{2}, 1] \times [0, 1) \to \mathbb{R}$ with the following properties*
(1) $H(x, \xi) = 0$ for $x \in [-\frac{1}{2}, 0]$
(2) $H(x, \xi) = \phi(\frac{\xi}{\psi(x)})$ for $x \in (0, 2\epsilon]$, where $\epsilon > 0$ is chosen for the definition of γ.
(3) $H(x, 0) = 0$ for $x \in [-\frac{1}{2}, 0]$ and $H(x, 0) = -1$ for $x \in (0, 1]$
(4) $H(1 + x, \xi) = H(x, \xi) + (\xi - 1)$ for $x \in [-\frac{1}{2}, 0]$
(5) The function $e^{2\pi i H(x,\xi)} : [-\frac{1}{2}, 0] \times \mathbb{T} \to \mathbb{C}$ is of class C^∞ away from $(0, 0)$ and $(1, 0)$

Proof. The proof is by construction, namely if $(x, \xi) \in [-\frac{1}{2}, 0] \times [0, 1)$ then

$$H(x, \xi) := \gamma(\frac{x}{2})\mathbb{1}_{(0,1]}(x)\phi(\frac{\xi}{\psi(x)}) + (1 - \gamma(\frac{x}{2}))\mathbb{1}_{(0,1]}(x)(\xi - 1) \tag{31}$$

satisfies properties (1) through (5). □
The regions of qualitatively different behavior for H are illustrated in the following figure: We then set $h(x, \xi) = e^{2\pi i H(x,\xi)}$

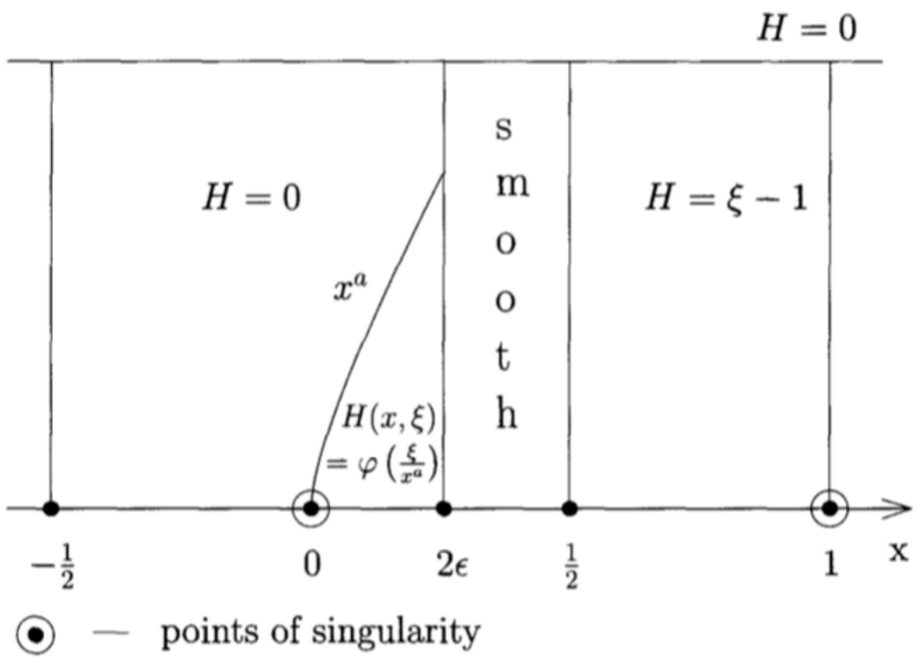

Fig. 1. Figure from [4] illustrating behavior of the constructed H function

for $(x, \xi) \in [-\frac{1}{2}, 1] \times \mathbb{T}$. Using property (4) of Lemma 3 we extend h to $\mathbb{R} \times \mathbb{T}$ via

$$h(k + x, \xi) = h(x, \xi)e^{2\pi i k(\xi - 1)} = h(x, \xi)e^{2\pi i k\xi} \tag{32}$$

Finally, the authors define g to be the inverse Zak transform of h via

$$g(k + x) = Z^{-1}(h)(k + x) \qquad k \in \mathbb{Z}, x \in [-\frac{1}{2}, \frac{1}{2}) \tag{33}$$

The authors then separate g into a singular part g_1 and a smooth part g_2 so that $g = g_1 + g_2$ and

$$\begin{aligned} g_1(k+x) &= \gamma(x)g(k+x) \\ g_2(k+x) &= (1-\gamma(x))g(k+x) \end{aligned} \tag{34}$$

Notice that h, being an exponential with purely imaginary exponent, immediately satisfies the criterion in Theorem 1 for $\mathcal{G}_{a,b}(g)$ to be an orthonormal basis for $L^2(\mathbb{R})$. Thus it remains estimate the integrals in Theorem 4. We rely heavily on the remark that followed Lemma 1 that

$$g(x-k) = (\mathcal{F}_\xi^{-1}h)(x,k) = (\mathcal{F}_\xi^{-1}h)(x-k,0) \tag{35}$$

to prove the following

Lemma 4. *g_2 belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$*

Proof. If $y \in \text{supp}(g_2)$, we have y of the form $y = k + x$ where $k \in \mathbb{Z}$ and $x \in [\epsilon, 1-\epsilon]$. Then

$$\begin{aligned} \partial_x^l g_2(y) = \partial_x^l g_2(k+x) &= \partial_x^l (1-\gamma(x))(\mathcal{F}_\xi^{-1}h)(x,-k) \\ &= \sum_{m=0}^{l} \binom{m}{l} \partial_x^{l-m}(1-\gamma(x))\partial_x^m(\mathcal{F}_\xi^{-1}h)(x,-k) \end{aligned} \tag{36}$$

Moreover, employing the dominated convergence theorem and integrating by parts we find

$$\begin{aligned} \partial_x^m(\mathcal{F}_\xi^{-1}h)(x,-k) &= \partial_x^m \int_{\mathbb{T}} e^{2\pi i k\xi} h(x,\xi)d\xi \\ &= \int_{\mathbb{T}} e^{2\pi i k\xi}\partial_x^m h(x,\xi)d\xi \\ &= \frac{1}{(2\pi i k)^n}\int_{\mathbb{T}} \partial_\xi^n e^{2\pi i k\xi}\partial_x^m h(x,\xi)d\xi \\ &= \frac{1}{(-2\pi i k)^n}\int_{\mathbb{T}} e^{2\pi i k\xi}\partial_\xi^n\partial_x^m h(x,\xi)d\xi \end{aligned} \tag{37}$$

Since the function h is smooth $[\epsilon, 1-\epsilon] \times \mathbb{T}$ we conclude that there exists $C_{l,n}$ such that

$$|\partial_x^l g_2(y)| \leq C_{l,n}\frac{1}{1+|k|^n} \tag{38}$$

Where again $y = k + x$. Since this bound is summable we conclude that $g_2 \in \mathcal{S}(\mathbb{R})$.□
Moving onto g_1 the authors of [4] employ the following form

$$g_1(k+x) = \gamma(x)(\delta(k) + \mathbb{1}_{[0,\frac{1}{2})}x^a F(kx^a)) \tag{39}$$

Where δ is the Kronecker delta and $F \in \mathcal{S}(\mathbb{R})$. Defining $\Phi(\xi) = e^{2\pi i\phi(\xi)} - 1$ this identity can be derived as

$$\begin{aligned} g_1(k+x) &= \gamma(x)(\mathcal{F}_\xi^{-1}h)(x,-k) = \gamma(x)\int_{\mathbb{T}} h(x,\xi)e^{2\pi i k\xi}d\xi \\ &= \gamma(x)\mathbb{1}_{[-\frac{1}{2},0]}\int_{\mathbb{T}} e^{2\pi i k\xi d\xi} + \gamma(x)\mathbb{1}_{[0,\frac{1}{2})}\int_{\mathbb{T}} e^{2\pi i\phi(\xi/x^a)}e^{2\pi i k\xi d\xi} \\ &= \gamma(x)\mathbb{1}_{[-\frac{1}{2},0]}\delta(k) + \gamma(x)\mathbb{1}_{[0,\frac{1}{2})}\left(\delta(k) + \int_{\mathbb{T}}(e^{2\pi i\phi(\xi/x^a)} - 1)e^{2\pi i k\xi})d\xi\right) \\ &= \gamma(x)\mathbb{1}_{[-\frac{1}{2},0]}\delta(k) + \gamma(x)\mathbb{1}_{[0,\frac{1}{2})}\left(\delta(k) + x^a\hat{\Phi}(-kx^a)\right) \\ &= \gamma(x)(\delta(k) + x^a F(kx^a)) \end{aligned} \tag{40}$$

Where $F(kx^a)$ is defined to be zero for $x \in [-\frac{1}{2}, 0]$. F is manifestly a Schwartz function here.□
We can improve the situation further by noting that $g_0(k+x) := g_1(k+x) - \gamma(x) = \gamma(x)\mathbb{1}_{[0,\frac{1}{2})}(x)x^a F(kx^a)$ for $k \in \mathbb{Z}$ and $x \in [-\frac{1}{2}, \frac{1}{2})$.

Theorem 5. *Let $a > 0$. If $A = 1 + 1/a$ and $\epsilon > 0$ then*

$$\int_{-\infty}^{\infty} |g(x)|^2 \frac{1+|x|^A}{\log^{1+\epsilon}(2+|x|)}dx < \infty \tag{41}$$

By Lemma 4 it is sufficient to replace g with g_0 in the first integral so that we have

$$\begin{aligned}\int_{-\infty}^{\infty}|g_0(x)|^2\frac{1+|x|^A}{\log^{1+\epsilon}(2+|x|)}dx &= \sum_{k\in\mathbb{Z}}\int_0^{1/2}|\gamma(x)x^aF(kx^a)|^2\frac{1+|k+x|^A}{\log^{1+\epsilon}(2+|x+k|)}dx\\ &\le C_1+C_2\sum_{|k|\ge 2}\int_0^{1/2}|x^aF(kx^a)|^2\frac{|k|^A}{\log^{1+\epsilon}(|k|)}dx\\ &= C_1+C_2\sum_{|k|\ge 2}\int_0^{(1/2)^a}|\frac{y}{k}F(y)|^2\frac{|k|^A}{\log^{1+\epsilon}(|k|)}|k|^{-\frac{1}{a}}\frac{1}{a}|y|^{\frac{1}{a}-1}dy\\ &\le C_1+C_2\sum_{|k|\ge 2}\frac{|k|^{-2+A-\frac{1}{a}}}{\log^{1+\epsilon}(|k|)a}\int_{\mathbb{R}}|y|^A|F(y)|^2dy\end{aligned} \tag{42}$$

The authors then use the fact that F was delicately constructed to be Schwartz class and that $A = 1 + 1/a$ so that we have

$$\int_{-\infty}^{\infty}|g_0(x)|^2\frac{1+|x|^A}{\log^{1+\epsilon}(2+|x|)}dx \le C_1+C_2\sum_{k\ge 2}\frac{1}{|k|\log^{1+\epsilon}(|k|)}<\infty \tag{43}$$

The inequality on the ξ side takes a similar form with $B = 1 + a$ as the weight exponent (I omit the full proof for sake of brevity) so that if $a = 1/(A-1) = B-1$ we come to the main theorem of [4] which is

Theorem 6. *If $\frac{1}{A}+\frac{1}{B}=1$ and $\epsilon > 0$ then there exists $g \in L^2(\mathbb{R})$ such that $\mathcal{G}_{a,b}(g)$ is an orthonormal basis for for $L^2(\mathbb{R})$ and such that*

$$\int_{-\infty}^{\infty}|g(x)|^2\frac{1+|x|^A}{\log^{1+\epsilon}(2+|x|)}dx<\infty \tag{44}$$

and

$$\int_{-\infty}^{\infty}|\hat{g}(\xi)|^2\frac{1+|\xi|^B}{\log^{1+\epsilon}(2+|\xi|)}d\xi<\infty \tag{45}$$

Theorem 4 is of course a corollary of this theorem for the case $A = B = 2$. □
Thus the Zak transform has provided us with a proof of the sharpness of the Balian Low theorem and in doing so a family of generating functions for $\mathcal{G}_{a,b}(g)$ which are optimal with respect to the uncertainty in Balian Low.

IV. Amalgam Balian Low

Thus we have seen that the Zak transform is intimately related to Gabor systems with $ab = 1$ via functional calculus and that it provides a means for analyzing the sharpness of the celebrated Balian Low uncertainty principle. It should be little surprise, therefore, that an interesting direction of modification of the Balian Low Theorem also arises from the Zak transform (I refrain from the word generalization because the results are in fact separate). Following [3] we introduce the Wiener amalgam space as

Definition 5. *(Wiener Amalgam Space). For $p, q \in \mathbb{Z}$ fixed define*

$$W(L^p,l^q) = \left\{ f : ||f||_{W(L^p,l^q)} = (\sum_{k\in\mathbb{Z}}||f\cdot \mathbb{1}_{[k,k+1]}||_p^q)^{1/q}\right\} \tag{46}$$

The amalgam space simultaneously encodes both local ($L^p(I)$) and global ($l^q(\mathbb{Z})$ information. Moreover, owing to part (i) of Lemma 2 we have that for $g \in W(L^p, l^q)$ the series Z_ag converges absolutely absolutely in $L^p(Q)$. Thus, the Zak transform maps continuously maps $W(L^p, l^q)$ into $L^p(Q)$.[3] The case of $p = \infty$ provides the following uncertainty theorem (note that the orthonormal basis criterion of Balian Low is relaxed to that of an exact frame):

Theorem 7. *(Amalgam Balian Low). Let $g \in L^2(\mathbb{R})$ and $ab = 1$. If $\mathcal{G}_{a,b}(g)$ is an exact frame for $L^2(\mathbb{R})$ then*

$$g \notin W(C_0,l^1) \text{ and } \hat{g} \notin W(C_0,l^1) \tag{47}$$

Proof. Without loss of generality the authors take $a = b = 1$ and proceed by contradiction. Namely, if $g \in W(C_0, l^1)$ then $g(x-k)e^{2\pi ik\xi}$ is continuous for each k. However, we note that

$$\begin{aligned}||Z_ag||_{L^p(Q)} &\le \sum_{k\in\mathbb{Z}}||g(x-k)e^{2\pi ik\xi}||_{L^p(Q)}\\ &= \sum_{k\in\mathbb{Z}}||g\cdot\mathbb{1}[k,k+1]||_{L^p(\mathbb{R})} = ||g||_{W(L^p,l^1)}<\infty\end{aligned} \tag{48}$$

Hence $Z_a g$ converges in L^∞ (uniformly) on Q, implying in particular that $Z_a g$ is continuous on $\mathbb{R}\times\mathbb{R}$. Therefore, by part *(ii)* of Lemma 2 $Z_a g$ must have a zero, so that the Gabor system $\mathcal{G}_{a,b}(g)$ cannot be a frame by Theorem 1. Since it is readily verified that $\hat{g_{m,n}} = e^{2\pi imn}g_{-n,m}$ it is the case that the Gabor system generated by g is a frame if and only if that corresponding to $\hat{g}$ is also a frame. Therefore the theorem holds. □

Remark The Amalgam Balian Low theorem is not a generalizaton of the Balian Low theorem, owing to the following example:

Lemma 5. *There exists g such that $g, \hat{g} \in W(C_0, l^1)$ while nevertheless satisfying $||xg(x)||_2||\xi\hat{g}(\xi)||_2 = \infty$.*

Proof. The proof in [3] is by explicit construction. Let $f : [0,1] \to \mathbb{R}$ be given as

$$f(t) = \max\{1 - |2t-1|, 0\} \tag{49}$$

Then f is continuous, piecewise linear and supported on $[0,1]$. If we take

$$g(t) = \sum_{n=1}^{\infty} n^{-3/2} f(t-n) \tag{50}$$

Then $g \in W(C_0, l^1)$ and moreover

$$\begin{aligned}\hat{g}(\xi) &= \sum_{n=1}^{\infty} n^{-3/2} e^{2\pi in\xi}\hat{f}(\xi) \\ &= (\sum_{n=1}^{\infty} n^{-3/2} e^{2\pi in\xi}) 2e^{-\pi i\xi}(\frac{\sin(\pi\xi/2}{\pi\xi})^2\end{aligned} \tag{51}$$

Employing the Fejer kernel in the last step. Because of the rapid decay of its Fourier coefficients, $\sum_{n=1}^{\infty} n^{-3/2}e^{2\pi in\xi}$ is a continuous, periodic function. Hence we obtain $\hat{g} \in W(C_0, l^1)$ also. The last step is to show the divergence of the following integral

$$\begin{aligned}\int_n^{n+1} |xg(x)|^2 dx &= n^{-3}\int_n^{n+1} |xg(x-n)|^2 dx \\ &\geq n^{-3}\int_n^{n+1} |ng(x-n)|^2 dx = n^{-1}||g||_2^2\end{aligned} \tag{52}$$

Hence

$$||xg(x)||^2 = \sum_{n=1}^{\infty}\int_n^{n+1} |xg(x)|^2 dx \geq ||f||_2^2 \sum_{n=1}^{\infty}\frac{1}{n} = +\infty \tag{53}$$

Thus take g to be the construction in the lemma.□

V. Concluding Remarks

Current areas of research involving the Zak transform include the theory of coherent states in quantum mechanics[**?**], sampling theorems for wavelet spaces[8], and analysis of non-stationary signals. I hope the collection of results herein demonstrate the unique role played by the Zak transform in understanding mixed time-frequency analysis, much the same way that the Fourier transform plays a unique role in understanding separated time and frequency analysis. The theory of the Zak transform is comparatively under developed and likely will provide many fruitful further avenues of research, some along the lines presented herein.

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